Coordinate Conversion Equation of Universal Transverse Mercator with Derivation

By Kacchan

1. Introduction

Universal Transverse Mercator (“UTM” afterwards) is a projection method for medium scale map, and is used for the maps (from 1:10000 to 1:200000) by Japan’ s Geospatial Information Authority. This paper describes the coordinate conversion equation from longitude and latitude to X and Y, the reason why the scale-coefficient is 0.9996, and the derivation of the coordinate conversion equation.

2. Coordinate Conversion Equation from longitude and latitude to X and Y

On UTM, latitude L_a, longitude L_o, and height H are converted to the world coordinate system ${}^t(X \quad Y \quad Z)$ by the equation below. Fig. 2.1 shows conceptual diagram of the world coordinate system.

$$\begin{pmatrix} X\\ Y\\ Z\end{pmatrix} \cong \begin{pmatrix} k_o\left\{aL_{o\_basis} + N(L_a)\left(\cos L_a\,(L_o - L_{o\_basis}) + \frac{1}{6}(\cos L_a)^3(1-(\tan L_a)^2 + e'^2(\cos L_a)^2)(L_o - L_{o\_basis})^3\right)\right\} \\ k_o\left\{S(L_a) + N(L_a)\left(\frac{1}{4}\sin(2L_a)\,(L_o-L_{o\_basis})^2 + \frac{1}{24}(\sin L_a\,(\cos L_a)^3)(5-(\tan L_a)^2+9e'^2(\cos L_a)^2+4e'^4(\cos L_a)^4)(L_o-L_{o\_basis})^4\right)\right\} \\ H \end{pmatrix}$$

Each coefficient represents:

L_a : latitude (unit:radian)
L_o : longitude (unit:radian)
H : height (unit:meter)
θ : longitude belt for converting (unit:degree) On UTM, fixed. ($\theta = 6[°]$)
$L_{o\_basis} = \left(\theta \times \left[\frac{1}{\theta}\left(\frac{180}{\pi}L_o\right)\right] + \frac{\theta}{2}\right) \times \frac{\pi}{180}$: basis-meridian for converting (unit:radian) [] represents “Gauss symbol” .
a : long radius of the globe (unit:meter) See also: Chart 2.1.
b : short radius of the globe (unit:meter) See also: Chart 2.1.
$e = \sqrt{1-\frac{b^2}{a^2}}$: first eccentricity
$e' = \sqrt{\frac{e^2}{1-e^2}}$: second eccentricity
k_o : scale-coefficient of projection origin (On UTM, fixed. ($k_o = 0.9996$) The reason will be discussed later.)
$R(L_a) = \frac{a(1-e^2)}{(1-e^2(\sin L_a)^2)^{\frac{3}{2}}}$: curvature radius of meridian (unit:meter)
$N(L_a) = \frac{a}{\sqrt{1-e^2(\sin L_a)^2}}$: curvature radius of prime vertical (unit:meter)
$S(L_a) = a\left(\left(1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}\right)L_a - \left(\frac{3e^2}{8}+\frac{3e^4}{32}+\frac{45e^6}{1024}\right)\sin(2L_a) + \left(\frac{15e^4}{256}+\frac{45e^6}{1024}\right)\sin(4L_a) - \frac{35e^6}{3072}\sin(6L_a)\right)$: meridional parts (unit:meter) The derivation of approximation is shown in reference [2].

Chart 2.1 Example of long radius and short radius (See also: reference [1])

	WGS84 Ellipsoid	GRS80	Bessel
a[m]	6378137.00000000	6378137.00000000	6377397.15500000
b[m]	6356752.31424518	6356752.31414035	6356078.96300000

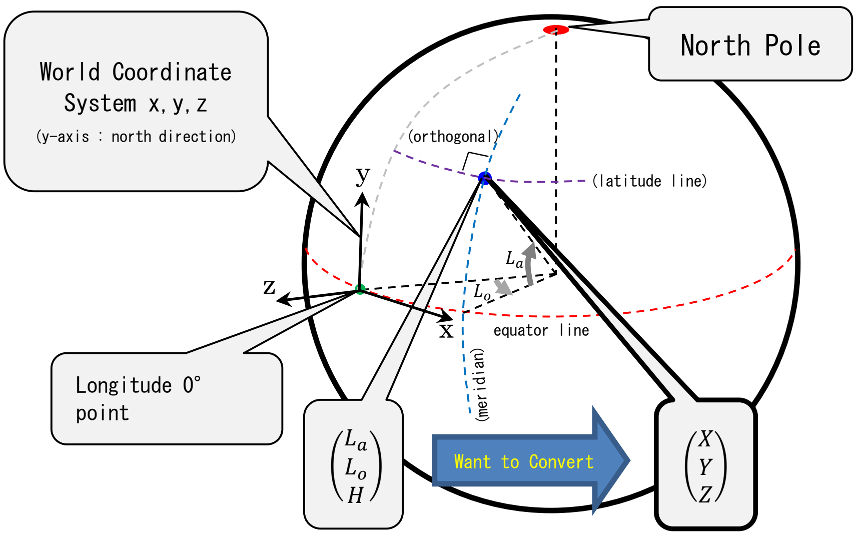


Fig 2.1 Conceptual Diagram of the World Coordinate System

3. Why scale-coefficient is 0.9996

Cylindrical projection means plotting each point on the globe onto the cylindroid around the globe, and UTM is one example of them.

First, regard the globe as $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1$ ellipsoid. Second, suppose the $\frac{x^2}{(k_o a)^2}+\frac{z^2}{(k_o b)^2}=1$ cylindroid around the globe (k_o is the scale-coefficient of projection origin). Then, every point on the ellipsoid is plotted onto the cross-point of the line from origin to the point and the surface of cylindroid. On UTM, one area per one projection is a thin sixtieth strip of the globe's surface in order to decrease error of projection. For simple explanation, the thin sixtieth strip is drawn as -3° ~+3° in Fig.3.1, but in practice the 60 sixtieth strips are 0° ~+3°, +3° ~+6°, …, and 354° ~360°.

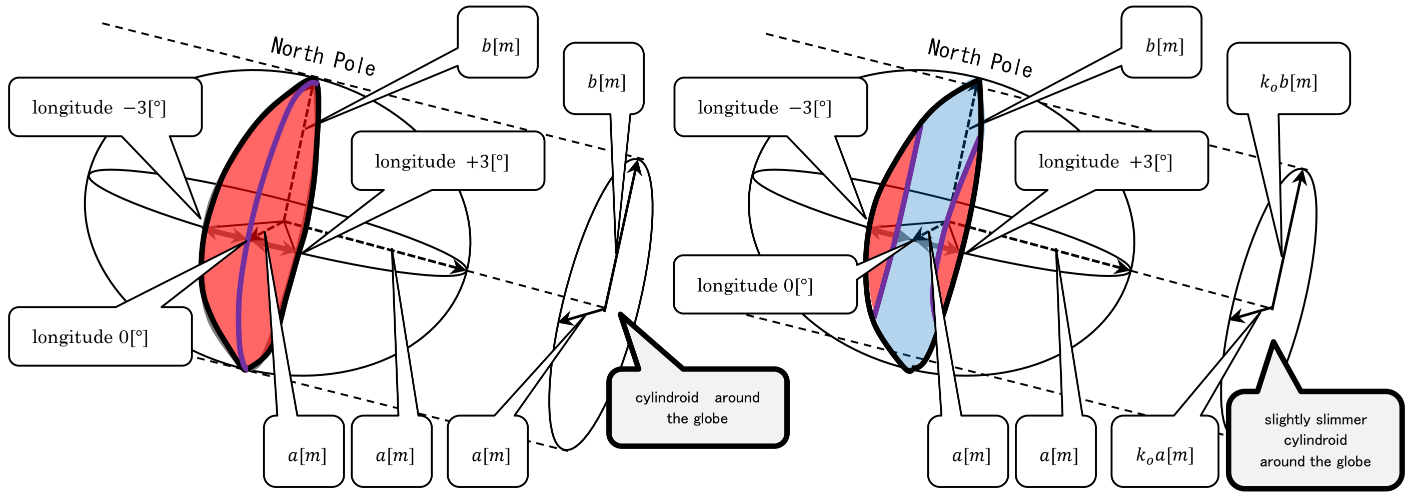
If $k_o = 1$ as shown in Fig.3.1(a), the cylindroid and ellipsoid contact each other only on where longitude is just 0° or 180°, so the scale is just 1.0 there. Otherwise the scale is more than 1.0, because on any other places the cylindroid is larger than the ellipsoid. Such is not ideal, supposing the ideal average scale over the whole strip should be 1.0.

In order to avoid this, the scale-coefficient should be slightly smaller as shown in Fig.3.1(b): $k_o = 0.9996$. This means narrowing the cylindroid. Therefore, there are two areas where the cylindroid is larger than the ellipsoid and where the ellipsoid is larger than the cylindroid, so the average scale becomes approximately 1.0.

Now, if we express the scale value where longitude is L_o and latitude is L_a as $M(L_o, L_a)$, $M(L_o, L_a) = \frac{(norm\ from\ the\ origin\ to\ surface\ of\ ellipsoid)}{(norm\ from\ the\ origin\ to\ surface\ of\ cylindroid)} = \frac{p'}{p}$. Considering $\frac{(p\cos L_a \cos L_o)^2}{a^2}+\frac{(p\cos L_a \sin L_o)^2}{a^2}+\frac{(p\sin L_a)^2}{b^2}=1$ and $\frac{(p'\cos L_a \cos L_o)^2}{(k_o a)^2}+\frac{(p'\sin L_a)^2}{(k_o b)^2}=1$, the scale value can be expressed as below:

$$M(L_o, L_a) = \frac{p'}{p} = k_o \frac{\sqrt{b^2(\cos L_a)^2 + a^2(\sin L_a)^2}}{\sqrt{b^2(\cos L_a \cos L_o)^2 + a^2(\sin L_a)^2}}$$

Fig.3.2 expresses the $M(L_o, L_a)$ value where longitude is L_o and latitude is L_a. On Fig.3.2(a) $k_o = 1$ and on Fig.3.2(b) $k_o = 0.9996$. The average scale in the $-3° \le L_o \le +3°$ area on the latter is nearer 1.0 than that on the former.



(a) scale-coefficient $k_o = 1$ (b) scale-coefficient $k_o = 0.9996(< 1)$

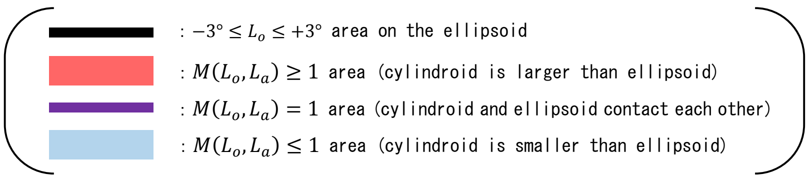
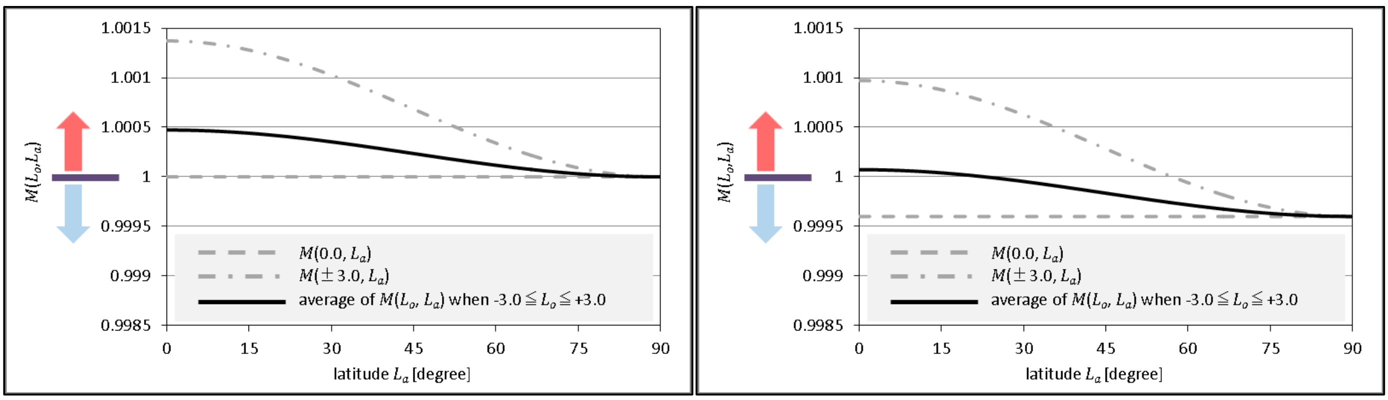


Fig.3.1 Conceptual Diagram of UTM



(a) scale-coefficient $k_o = 1$ (b) scale-coefficient $k_o = 0.9996 < 1$

Fig.3.2 graph of scale value $M(L_o, L_a)$

4. Derivation

Derivation of coordinate conversion equation is below. (See also: reference [3])

First, meridional parts (arc length from equator to the target (L_o, L_a)) can be calculated by $\int_{L'_a=0}^{L'_a=L_a} R(L'_a)\,dL'_a$. In order to calculate both X coordinate and Y coordinate from this value all at once, this paper regard orthogonal coordinate system as complex plane, so the X coordinate means real part and the Y coordinate means imaginary part, respectively. However, (L_o, L_a) plane is not orthogonal coordinate system, because of curved surface. So, this paper use z instead of L_a.

Fig.4.1 shows conceptual diagram of z. The dimension of z is [radian], and always orthogonal to L_0. The z value can be calculated by the integral equation below, because $\frac{R(L'_a)\cdot\Delta L'_a}{N(L'_a)\cos L'_a\cdot\Delta L'_o}\cdot\Delta L'_o$ is added to z as the longitude increases by $\Delta L'_o$.

$$z=\sum\frac{R(L'_a)\cdot\Delta L'_a}{N(L'_a)\cos L'_a\cdot\Delta L'_o}\cdot\Delta L'_o=\sum\frac{R(L'_a)\Delta L'_a}{N(L'_a)\cos L'_a}=\int_{L'_a=0}^{L'_a=L_a}\frac{R(L'_a)}{N(L'_a)\cos L'_a}dL'_a$$

Now, X and Y value can be calculated by the equation below. In the equation, $\int_{L'_a=0}^{L'_a=L_a} R(L'_a)\,dL'_a$ is expressed as a function of complex z, and $z=u+vi$ (real part u expresses X direction, imaginary part v expresses Y direction). X value is regarded as real part of $Q(z)$, and Y value is regarded as imaginary part of $Q(z)$.

$$\int_{L'_a=0}^{L'_a=L_a} R(L'_a)\,dL'_a=Q(z)=Q(u+vi)$$

By partial differentiation, $\frac{\partial z}{\partial L_a}=\frac{R(L_a)}{N(L_a)\cos L_a}$, which can be also expressed as $N(L_a)\cos L_a\,dz=R(L_a)dL_a$. So,

$$\int_{L'_a=0}^{L'_a=L_a} R(L'_a)\,dL'_a=\int_{z'=0}^{z'=z}N(L'_a)\cos L'_a\,dz'=Q(z)=Q(u+vi)$$

This equation means that meridional parts can be expressed as function of complex z.

By performing Taylor expansion near $z=vi$,

$$\begin{aligned}\int_{L'_a=0}^{L'_a=L_a} R(L'_a)\,dL'_a&=\int_{z'=0}^{z'=z}N(L'_a)\cos L'_a\,dz'=Q(u+vi)=Q(vi)+\frac{u}{1!}\frac{\partial Q(vi)}{\partial z}+\frac{u^2}{2!}\frac{\partial^2Q(vi)}{\partial z^2}+\frac{u^3}{3!}\frac{\partial^3Q(vi)}{\partial z^3}+\frac{u^4}{4!}\frac{\partial^4Q(vi)}{\partial z^4}+\frac{u^5}{5!}\frac{\partial^5Q(vi)}{\partial z^5}+\cdots\\&=iQ(v)+i\cdot\left(\frac{1}{i}\right)^1\frac{u}{1!}\frac{\partial Q(v)}{\partial z}+i\cdot\left(\frac{1}{i}\right)^2\frac{u^2}{2!}\frac{\partial^2Q(v)}{\partial z^2}+i\cdot\left(\frac{1}{i}\right)^3\frac{u^3}{3!}\frac{\partial^3Q(v)}{\partial z^3}+i\cdot\left(\frac{1}{i}\right)^4\frac{u^4}{4!}\frac{\partial^4Q(v)}{\partial z^4}+i\cdot\left(\frac{1}{i}\right)^5\frac{u^5}{5!}\frac{\partial^5Q(v)}{\partial z^5}+\cdots\end{aligned}$$

By separating imaginary part from real part, multiplying k_o, supposing $u=L_o-L_{o\_basis}$, adding $k_oaL_{o\_basis}$ to X in order to shift the basis of Taylor expansion, and considering $\frac{\partial Q(z)}{\partial z}=N(L_a)\cos L_a$, X and Y value can be expressed as below. For detailed way of approximation, see reference [2] and [3].

$$\begin{aligned}\begin{pmatrix}X\\Y\end{pmatrix}&=\begin{pmatrix}k_oaL_{o\_basis}\\0\end{pmatrix}+\begin{pmatrix}k_o\cdot Re(Q)\\k_o\cdot Im(Q)\end{pmatrix}=\begin{pmatrix}k_oaL_{o\_basis}\\0\end{pmatrix}+k_o\begin{pmatrix}\frac{u}{1!}\frac{\partial Q(v)}{\partial z}-\frac{u^3}{3!}\frac{\partial^3Q(v)}{\partial z^3}+\frac{u^5}{5!}\frac{\partial^5Q(v)}{\partial z^5}+\cdots\\Q(v)-\frac{u^2}{2!}\frac{\partial^2Q(v)}{\partial z^2}+\frac{u^4}{4!}\frac{\partial^4Q(v)}{\partial z^4}+\cdots\end{pmatrix}\\&=\begin{pmatrix}k_oaL_{o\_basis}\\0\end{pmatrix}+k_o\begin{pmatrix}N(L_a)\cos L_a\,(L_o-L_{o_{basis}})-\frac{(L_o-L_{o_{basis}})^3}{3!}\frac{\partial^3Q(v)}{\partial z^3}+\frac{(L_o-L_{o_{basis}})^5}{5!}\frac{\partial^5Q(v)}{\partial z^5}+\cdots\\\int_{L'_a=0}^{L'_a=L_a}R(L'_a)\,dL'_a-\frac{(L_o-L_{o_{basis}})^2}{2!}\frac{\partial^2Q(v)}{\partial z^2}+\frac{(L_o-L_{o_{basis}})^4}{4!}\frac{\partial^4Q(v)}{\partial z^4}+\cdots\end{pmatrix}\\&\cong k_o\begin{pmatrix}aL_{o\_basis}+N(L_a)\left(\cos L_a\,(L_o-L_{o\_basis})+\frac{1}{6}(\cos L_a)^3(1-(\tan L_a)^2+e'^2(\cos L_a)^2)(L_o-L_{o\_basis})^3\right)\\S(L_a)+N(L_a)\left(\frac{1}{4}\sin(2L_a)\,(L_o-L_{o\_basis})^2+\frac{1}{24}(\sin L_a\,(\cos L_a)^3)(5-(\tan L_a)^2+9e'^2(\cos L_a)^2+4e'^4(\cos L_a)^4)(L_o-L_{o\_basis})^4\right)\end{pmatrix}\end{aligned}$$

And the meridional parts can be approximated as below. (See also: reference [2])

$$\int_{L'_a=0}^{L'_a=L_a}R(L'_a)\,dL'_a\cong a\left(\left(1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}\right)L_a-\left(\frac{3e^2}{8}+\frac{3e^4}{32}+\frac{45e^6}{1024}\right)\sin(2L_a)+\left(\frac{15e^4}{256}+\frac{45e^6}{1024}\right)\sin(4L_a)-\frac{35e^6}{3072}\sin(6L_a)\right)=S(L_a)$$

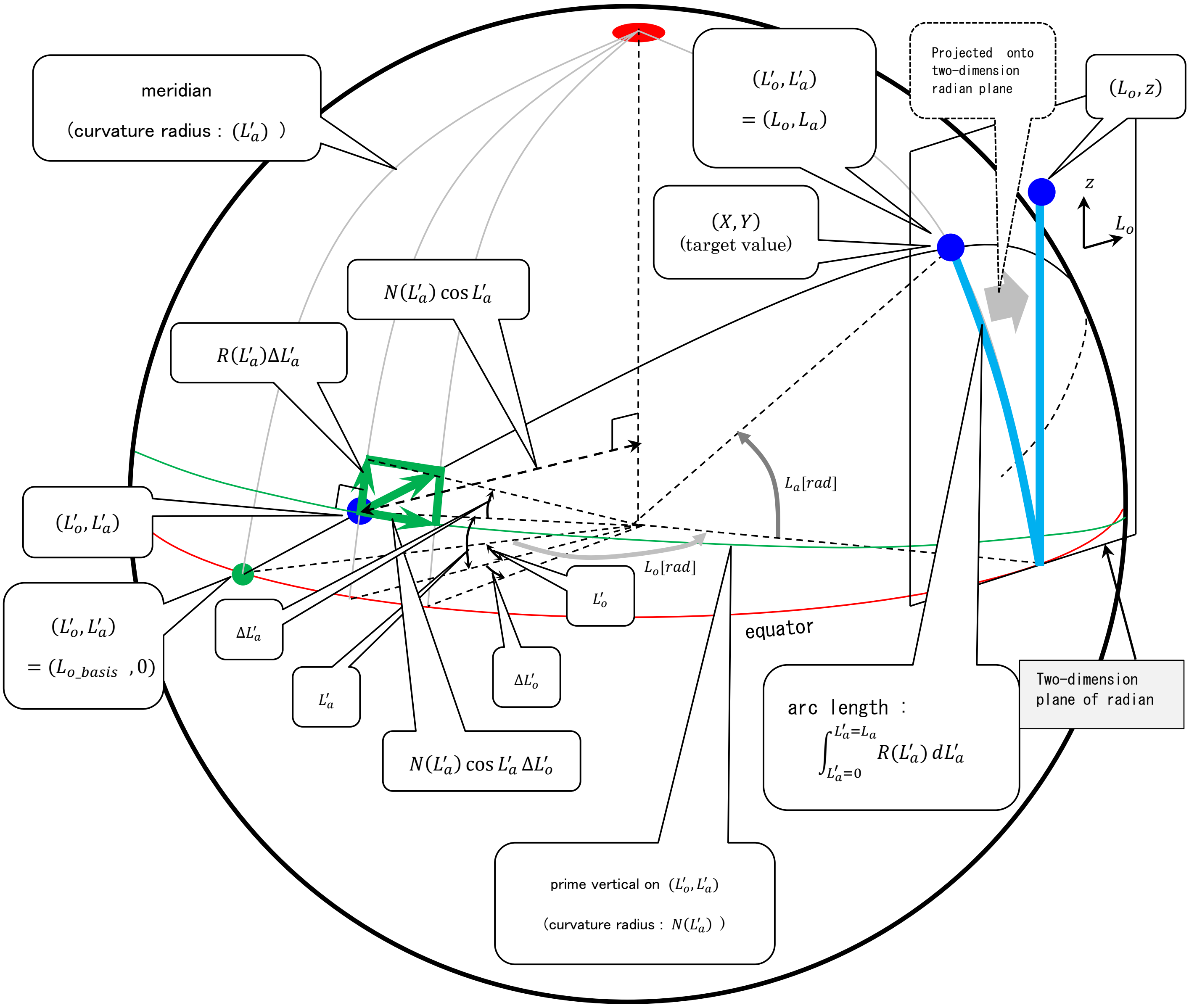


Fig. 4.1 L_o, L_a,and orthogonal value z

5. Reference

[1] http://www.uwgb.edu/dutchs/usefuldata/utmformulas.htm

[2] http://w01.tp1.jp/~a540015671/program/utm.pdf

[3] Paul D. Thomas, “Conformal Projections in Geodesy and Cartography”, U.S. Department of Commerce, Special Publication No.251, 1952